EXTENSION OF COLLATZ CONJECTURE AND A PROOF OF COLLATZ-1 CONJECTURE AND FALSIFICATION OF COLLATZ-5 AND COLLATZ-181 CONJECTURES

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Abstract. The *Collatz conjecture* is extended into *Collatz-M conjecture*, and a proof of *Collatz-1 conjecture* and falsification of *Collatz-5 and Collatz-181 conjecturesis* are given in this paper.

1. Introduction of the Collatz conjecture

Quote from [1], "Let $\mathbb{N} := \{0, 1, 2, \dots\}$ denote the natural numbers, so that $\mathbb{N} + 1 = \{1, 2, 3, \dots\}$ are the positive integers. The *Collatz map* $\mathrm{Col}\colon \mathbb{N}+1 \to \mathbb{N}+1$ is defined by setting $\mathrm{Col}(N) := 3N + 1$ when N is odd and $\mathrm{Col}(N) := N/2$ when N is even. For any $N \in \mathbb{N}+1$, let $\mathrm{Col}_{\min}(N) := \min \mathrm{Col}^{\mathbb{N}}(N) = \inf_{n\in\mathbb{N}} \mathrm{Col}^n(N)$ denote the minimal element of the Collatz orbit $\mathrm{Col}^{\mathbb{N}}(N) := \{N, \mathrm{Col}(N), \mathrm{Col}^2(N), \dots\}$. We have the infamous *Collatz conjecture* (also known as the $3x + 1$ conjecture):

Conjecture 1.1 (Collatz conjecture)**.** *We have* $\mathrm{Col}_{\min}(N) = 1$ *for all* $N \in \mathbb{N} + 1$*."*

2. Extension of Collatz conjecture–Collatz-M conjecture

The Collatz conjecture can be extended to the *Collatz-M conjecture*, as follows.

Let $\mathbb{N} := \{0, 1, 2, \dots\}$ denote the natural numbers, so that $\mathbb{N} + 1 = \{1, 2, 3, \dots\}$ are the positive integers. The *Collatz-M map*, where M is odd and $M \in \mathbb{N} + 1$, $\mathrm{Col}_M\colon \mathbb{N} + 1 \to \mathbb{N} + 1$ is defined by setting $\mathrm{Col}_M(N) := MN + 1$ when N is odd, and $\mathrm{Col}_M(N) := N/2$ when N is even. For any $N \in \mathbb{N} + 1$, let $\mathrm{Col}_{M_{\min}}(N) := \min \mathrm{Col}_M^{\mathbb{N}}(N) = \inf_{n\in\mathbb{N}} \mathrm{Col}_M^n(N)$ denote the minimal element of the Collatz-M orbit $\mathrm{Col}_M^{\mathbb{N}}(N) := \{N, \mathrm{Col}_M(N), \mathrm{Col}_M^2(N), \dots\}$. We have the *Collatz-M conjecture*:

Conjecture 2.1 (Collatz-M conjecture)**.** $\mathrm{Col}_{M_{\min}}(N) = 1$*, where* M *is odd and* $M \in \mathbb{N} + 1$*, for all* $N \in \mathbb{N} + 1$*.*

Therefore, from **Conjecture 2.1**, the *Collatz conjecture* is the *Collatz-3 conjecture.*

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3. Proof of Collatz-1 conjecture

From **Conjecture 2.1**, when $M = 1$, we have $\mathrm{Col}_1(N) := N + 1$ when N is odd, and $\mathrm{Col}_1(N) := N/2$ when N is even. Therefore we have

Conjecture 3.1 (Collatz-1 conjecture)**.** $\mathrm{Col}_{1_{\min}}(N) = 1$ *for all* $N \in \mathbb{N} + 1$.

Proof. All integers can be devided into two groups: odd integers and even integers, and an odd integer +1 is an even integer.

Even integers can be divided into two groups: power of 2 integers and not power of 2 even integers.

Case 1 For even integers N which are power of 2, since $\mathrm{Col}_0(N) := N/2$, let $N = 2^k, k \geq 1, k \in \mathbb{N}$, then after k mappings, we arrive the minimal element of Collatz-1 orbit, i.e., $\mathrm{Col}_{1_{\min}}(N) = 1$ for all power of 2 integers $N \in \mathbb{N} + 1$.

Case 2 For even integers N which are not power of 2, since $\mathrm{Col}_1(N) := N/2$, let $N = 2^l G, l \geq 1, l \in \mathbb{N}$, $G \in \mathbb{N} + 1$ and G is odd. After l mappings, we get $\mathrm{Col}_1(N) = G$, then $\mathrm{Col}_1(G) := G + 1$. Let $G = 2Y + 1$ and Y is odd, then $\mathrm{Col}_1(G + 1) = Y + 1$.

If $Y + 1 = 2^n H, n \geq 1, n \in \mathbb{N}$, H is odd, it returns to **Case 2**;

If $Y + 1 = 2^m, m \geq 1, m \in \mathbb{N}$, according to **Case 1**, we have $\mathrm{Col}_{1_{\min}}(N) = 1$.

Therefore, $\mathrm{Col}_{1_{\min}}(N) = 1$ for all even $N \in \mathbb{N} + 1$.

Case 3 For odd integers N, $N + 1$ are even integers. Therefore from **Case 1** and **Case 2**, we can arrive the finial result:

$\mathrm{Col}_{1_{\min}}(N) = 1$ for all $N \in \mathbb{N} + 1$. □

It is noticed that the results of *Collatz-1 map* are in general almost monotonic descent (for N is odd, $\mathrm{Col}_M(N) := N + 1$), which makes the *Collatz-1 conjecture* provable.

4. Falsification of Collatz-5 and Collatz-181 conjectures

For $M = 5$, $N = 5$, it exists a periodic orbit which dose not include 1, i.e.,

$$26, 13, 66, 33, 166, 83, 416, 208, 104, 52, 26$$

and for $M = 181$, $N = 27$, it exists a periodic orbit which dose not include 1, i.e.,

$$27, 4888, 2444, 1222, 611, 110592, 55296, 27648, 13824, 6912, 3456, 1728, 864,$$
$$432, 216, 108, 54, 27$$

Therefore the *Collatz-5 conjecture* and the *Collatz-181 conjecture* are false.

References

[1] T. Tao, *Almost all orbits of the Collatz map attain almost bounded values*, arXiv: 1909.03562v2 [math.PR] 13 Sept. 2019.

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